Proof of the Witten-Yau Conjecture

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The Witten-Yau theorem in the AdS/CFT correspondence conjectures that the conformal boundary to AdS space must possess a metric of non-negative scalar curvature for the conformal field theory defined thereon to be free of pathologies. By employing various tools from conformal geometry - such as almost Einstein structures, collapsing sphere products and tractor bundles - we rigorously prove this conjecture.

I. INTRODUCTION

The *Witten-Yau theorem* in the AdS/CFT correspondence states that when the conformal boundary of an asymptotically hyperbolic spacetime possesses a metric of non-negative scalar curvature, then it is necessarily connected and all integral homologies vanish. The conjectured existence of a non-negative scalar curvature metric on the conformal boundary is essential, since it has been shown that the conformal field theory (CFT) defined thereon is unstable otherwise[1]. It is therefore of great importance to verify the existence of such non-negative scalar curvature boundary metrics - for in their absence, there would simply be no AdS/CFT correspondence[24]. For brevity, we refer to this conjecture underlying the Witten-Yau theorem as the *Witten-Yau conjecture* and present in this article a rigorous proof. Precisely: we employ various techniques from modern conformal geometry to show that every asymptotically hyperbolic Euclidean spacetime possesses a conformal boundary metric of non-negative scalar curvature.

By way of introduction, let us examine the most notable physical pathology stemming from the absence of a non-negative scalar curvature boundary metric. Let $(M, [g])$ be an n-manifold endowed with a conformal class of metrics $[g]$ - hereafter referred to as a *conformal structure* for M - and consider the conformally-invariant operator

$$\Delta' = \Delta + \frac{n-2}{4(n-1)}\mathsf{R},$$

where $\Delta = -\nabla_a \nabla^a$ is the Laplace-Beltrami operator and R the scalar curvature of some metric g on M. Witten & Yau showed that if Δ' is positive definite then a particular string theory action related to the CFT is bounded below ([1],[2]). However, if Δ' admits a negative eigenvalue then the string action is unbounded below, rendering the CFT unstable.

The class of manifolds pertinent to the AdS/CFT correspondence are the so-called *Poincaré-Einstein* manifolds ([3]-[5], and so it is to these manifolds that we construct a conformal boundary metric of non-negative scalar curvature, thereby proving the conjecture.

Our presentation is as follows. We firstly introduce some background from modern conformal geometry. After defining basic concepts such as conformal structures on manifolds, we discuss a special fibre bundle over the conformal manifold called the 'tractor bundle' which controls the conformal geometry. The tractor bundle admits a workable calculus, and significant progress in both modern conformal geometry ([6]-[13]) and theoretical physics ([14]-[17]) has been made through the use of tractor methods. After formalising these notions, we discuss an important generalisation of the concept of an Einstein metric called an *almost Einstein metric* - of which Poincaré-Einstein metrics are a particular case. Recall that an *Einstein metric* g_{ab} on a manifold M is one whose Ricci tensor is proportional to the metric

$$\mathsf{R}_{ab} = \lambda g_{ab}$$

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where λ is, in theoretical physics, called the *cosmological constant*[18]. Given a metric g^o on the interior M^o of a manifold M, then g^o is *conformally compact* if it may be written as $g^o = \sigma^{-2} g$, where σ is a conformal scale (to be defined later.) Such metrics are called *Poincaré-Einstein metrics* if they are also Einstein, and it is these which are of physical pertinence in the AdS/CFT Correspondence. Poincaré-Einstein are *asymptotically hyperbolic*, meaning that their sectional curvature tends to -1 at the boundary. Generalising this concept, we say that the metric $g^o = \sigma^{-2} g$ is *almost Einstein* if it is Einstein on all open dense subsets where the conformal scale σ is non-vanishing. The set of points for which the conformal scale vanishes will be of particular interest to us, as it will be realised as a conformal infinity space (in the sense of Penrose[19],[20]) for an almost Einstein manifold with non-positive scalar curvature. After properly introducing these concepts, we will discuss Leitner's 'collapsing sphere product' construction ([21],[22]), and by extension it will be seen that manifolds admitting multiple almost Einstein structures are conformally equivalent to products of Poincaré-Einstein manifolds with spheres. The tools and standard results alluded to above allow us to prove the Witten-Yau conjecture for general Poincaré-Einstein manifolds.

II. CONFORMAL GEOMETRY & TRACTOR CALCULUS

The boundary manifold ∂M in the AdS/CFT correspondence is endowed with a 'conformal structure'. Heuristically, metrics on ∂M are only defined up to scale, and so the intrinsic geometry of ∂M is that of *conformal geometry*. In this section, we will recall the rudiments of conformal geometry required for proving the Witten-Yau conjecture. We work squarely within the Euclidean framework, where manifolds (and their boundaries) are Riemannian, rather than pseudo-Riemannian. For clarity, we will include or suppress tensorial indices as required. Let M denote an n-dimensional manifold and g a Riemannian metric. A *conformal structure*[23] is induced by a set of pairwise-homothetic Riemannian metrics

$$c \equiv [g] := \{\Omega^2 g \mid \Omega^2 \in \mathcal{F}^+\}$$

which are equivalent modulo a scaling by some positive function $\Omega^2 \in \mathcal{F}^+$, and we call the pairing (M, c) a *conformal manifold*.

Let us now introduce Penrose's *abstract index notation*. In this set-up, a single kernel letter is adorned with abstract indices. For example, denoting world-indices by lower case latin letters, a vector field V would be written as a section $V^a \in \Gamma(\mathcal{E}^a)$, where $\mathcal{E}^a$ is the abstractly-indexed tangent bundle and the notation $\Gamma(\cdot)$ denotes the space of sections. Likewise, a two-form ω would be written $\omega_{ab} \in \Gamma(\mathcal{E}_{[ab]})$ with $\mathcal{E}_{[ab]}$ the second exterior power of the cotangent bundle, and so on. We will use abstract index notation interchangeably with the standard tensorial notation depending on context.

This notation in place, let us consider a special class of line bundles of particular importance in conformal geometry: *conformal density bundles*. A particularly thorough elucidation is given in[6]. Let $CO(n) = SO(n) \ltimes \mathbb{R}$ denote the linear conformal group, and let $\mathcal{P}$ be the $CO(n)$-principal bundle defined by $[g]$. Paper[6] constructs a class of weighted line bundles, arising from the non-trivial centre of the conformal group. For some representation ρ, we define the line bundle $\mathcal{E}[w] := \mathcal{P} \times_\rho \mathbb{R}$ by

$$\rho(c)(x) = -w \frac{\det(c)}{2n} x, \tag{1}$$

where w is the conformal weight, c some conformal structure and $x \in \mathbb{R}$. Connections on the tangent bundle extend to connections on these weighted line bundles, and so we may write a conformally-weighted covector (say) ω of conformal weight w as $\omega_a \in \Gamma(\mathcal{E}_a[w])$ where $\mathcal{E}_a[w] := \mathcal{E}_a \otimes \mathcal{E}[w]$. In particular, sections of $\mathcal{E}[1]$ are called *conformal scales*. Now, each conformal structure $[g]$ determines a tautological section $\boldsymbol{g} \in \Gamma(\odot^2 T^* M[2])$ by

$$g \mapsto \boldsymbol{g} = \frac{\det g}{n} g,$$

and so we regard a conformal structure $[g]$ and the conformal metric $\boldsymbol{g}$ it determines as being equivalent.

Let us now introduce some concepts from Riemannian geometry. For some vector field V^d, the Riemann tensor $\mathsf{R}_{ab}{}^c{}_d$,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = \mathsf{R}_{ab}{}^c{}_d V^d,$$

may be written in terms of the conformal metric as

$$\mathsf{R}_{abcd} = \mathsf{W}_{abcd} + 2\boldsymbol{g}_{c[a}\mathsf{P}_{b]d} - 2\boldsymbol{g}_{d[b}\mathsf{P}_{c]a},$$

where W_{abcd} is the conformally invariant *Weyl tensor* and P_{ab} is the trace-adjusted Ricci tensor, or *Schouten tensor*

$$\mathsf{P}_{ab} = -\frac{1}{n-2}\left(\mathsf{R}_{ab} - \frac{1}{2n-2}\mathsf{R}\boldsymbol{g}_{ab}\right).$$

Over any conformal manifold there exists a rank $(n+2)$ vector bundle called the *standard tractor bundle* which, upon choosing a metric in the conformal class, may be written as a direct sum of conformal density bundles:

$$\mathcal{T} = \mathcal{E}^A \overset{g}{:=} \mathcal{E}[1] \oplus \mathcal{E}^a[1] \oplus \mathcal{E}[-1].$$

Sections of the both the tractor bundle and its various tensor powers are referred to simply as tractors, and adorned with upper case latin indices. In particular, sections of the standard tractor bundle are triples

$$\Gamma(\mathcal{T}) \ni V^A := \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix},$$

where σ is referred to as the *primary part*, μ^a the *secondary part* and so on. Moreover, there is an invariant signature $(n+1,1)$ bilinear form g_{AB} defined on $\mathcal{T}$ called the *tractor metric*, with associated *tractor inner product* $\langle\ ,\ \rangle$. In terms of two tractor vectors V^A and V^B

$$g_{AB}V^AV^B = \sigma\gamma + \mu^a\beta_a + \rho\alpha,$$

where

$$V^A = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}, \qquad V^B = \begin{pmatrix} \alpha \\ \beta^b \\ \gamma \end{pmatrix}. \tag{2}$$

Choosing a conformal scale $\sigma \in \Gamma(\mathcal{E}[1])$, the *tractor connection* $\nabla^{\mathcal{T}}_a$ on $\mathcal{T}$ is defined by

$$\nabla^{\mathcal{T}}_a \begin{pmatrix} \sigma \\ \mu^b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu^b + \delta_a{}^b\rho + \mathsf{P}_a{}^b\sigma \\ \nabla_a\rho - \mathsf{P}_{ac}\mu^c \end{pmatrix}$$

for some $V^B = (\sigma, \mu^b, \rho)^{\perp}$. Moreover, there is an operator called the *Thomas D-operator*

$$D_A : \mathcal{T}_{B\ldots C}[w] \to \mathcal{T}_{AB\ldots C}[w-1]$$

which may be used to differentiate conformally-weighted tractors, and acts on some tractor V_B according to

$$D_AV_B = \begin{pmatrix} (d+2w-2)wV_B \\ (d+2w-2)\nabla^{\mathcal{T}}_a V_B \\ (\Delta - w\mathsf{P})V_B. \end{pmatrix}.$$

We say that a tractor I is parallel if

$$\nabla^{\mathcal{T}}_a I = 0. \tag{3}$$

Parallel tractors play a special role in conformal geometry, since they are in one-to-one correspondence with Einstein metrics in the conformal class ([3]-[5],[8],[15],[16]). For later definitions, it is important to realise that the parallel tractor equation (3) is equivalent to the equation

$$\text{trace-free}(\nabla^g\nabla^g\sigma + \mathsf{P}^g\sigma) = 0, \tag{4}$$

where ∇^g is the Levi-Čività connection associated with the metric g. Likewise, equation (4) also characterises the existence of *almost Einstein structures*.

Lastly, we introduce *tractor p-forms*. Let $\mathcal{T}^*$ denote the dual tractor bundle and $\Lambda^p\mathcal{T}^*$ denote the bundle whose sections are p-forms on $\mathcal{T}$. It is a standard result in conformal geometry[21] that for $2 \leq p \leq n$, the tractor p-form bundle $\Lambda^p\mathcal{T}^*$ admits a direct sum decomposition

$$\Lambda^p\mathcal{T}^* \cong \Lambda^{p-1}T^*M[p] \oplus \big(\Lambda^p T^*M[p] \oplus \Lambda^{p-2}T^*M[p-2]\big) \oplus \Lambda^{p-1}T^*M[p-2],$$

with projection

$$\Pi : \Lambda^p\mathcal{T}^* \to \Lambda^{p-1}T^*M[p] \tag{5}$$

onto the bundle of $(p-1)$-forms of conformal weight p.

III. ALMOST EINSTEIN STRUCTURES

Following the presentation of[21], let us consider a conformal manifold (M^n, c) of dimension ≥ 3 with standard tractor $I \in \Gamma(\mathcal{T})$ whose primary part $\sigma = \Pi(I)$, where Π is the projector from (5). As remarked above, if the tractor I is parallel (with respect to the tractor connection), then the metric $g = \sigma^{-2}\boldsymbol{g}$ is Einstein. It was remarked in[21] however, that the conformal scale $\sigma = \Pi(I)$ of such an Einstein metric may have zeroes on M, and so g is Einstein only outwith the zero set of σ. This motivates the definition of a (directed) *almost Einstein structure*.

Theorem 1 [Gover-Leitner]
A (directed) almost Einstein structure is a conformal manifold (M^n, c) equipped with a parallel (standard) tractor $I \neq 0$. The mapping from non-trivial solutions of (4) to parallel tractors is by $\sigma \mapsto \frac{1}{n}D\sigma$ with inverse $I \mapsto \sigma := \Pi(I)$ and σ is non-vanishing on an open dense set $M\backslash\Sigma(\sigma)$. On this set $g := \sigma^{-2}\boldsymbol{g}$ is Einstein.

Herein, $\Sigma(\sigma)$ is termed the *scale singularity set* and $\sigma = \Pi(I) \in \Gamma(\mathcal{E}[1])$ an *almost Einstein scale*. It was shown in[3] that the structure of scale singularities sets is highly sensitive to the scalar curvature of the conformal manifold. Let us write the Ricci tensor for an Einstein manifold as $\mathsf{R}_{ab} = (n-1)Sg_{ab}$, where S is a constant. It is a fact in conformal geometry that the sign of the scalar curvature on an almost Einstein manifold is opposite to the sign of the tractor inner product, or in other words $S(\sigma) = -\langle I, I\rangle$ for $\sigma = \Pi(I)$ and $I \in \Gamma(\mathcal{T})$. Gover[3] has characterised the structure of the singularity sets according to the sign of the constant S:

Theorem 2 [Gover]
Let (M, c, I) be a Riemannian signature almost Einstein structure and $\sigma = \Pi(I)$. If $S(\sigma) > 0$ then $\Sigma(\sigma)$ is empty and $(M, \sigma^{-2}\boldsymbol{g})$ is Einstein with positive scalar curvature; If $S(\sigma) = 0$ then $\Sigma(\sigma)$ is either empty or consists of isolated points, and $(M\backslash\Sigma(\sigma), \sigma^{-2}\boldsymbol{g})$ is Ricci-flat; if $S(\sigma) < 0$ then the scale singularity set $\Sigma(\sigma)$ is either empty or else is a totally umbillic smooth hypersurface, and $(M\backslash\Sigma(\sigma), \sigma^{-2}\boldsymbol{g})$ is Einstein of negative scalar curvature. In the case when $\Sigma(\sigma) \neq \emptyset$, we have from[4] that $\Sigma(\sigma)$ is a conformal infinity in the sense of Penrose[19], and also that $\Sigma(\sigma)$ inherits the conformal structure $(\Sigma, [g]_\Sigma)$ when $S(\sigma) < 0$. Pertinent to our discussion is the case when a conformal manifold (M^n, c) admits multiple almost Einstein structures. To discuss this, let

$$\mathcal{P} := \{I \in \Gamma(\mathcal{T}) \mid \nabla^{\mathcal{T}} I = 0\} \tag{6}$$

denote the space of parallel tractors on a conformal manifold (M, c) and let $\mathcal{S} \subset \mathcal{P}$. Following[21], the intersection of the scale singularities of $\Pi(I)$ for $I \in S$ is given by

$$\Sigma(\mathcal{S}) := \bigcap_{I \in \mathcal{S}} \Sigma(\Pi(I)).$$

Moreover, from[21] we have that if two different hypersurface singularities of almost Einstein structures intersect, then they do so transversally, and so $\Sigma(\mathcal{S})$, like $\Sigma(\sigma)$, is a smooth submanifold of (M, c) for any subset $\mathcal{S} \subset \mathcal{P}$.

Having introduced $\Sigma(\mathcal{S})$ and commented that it is a smooth submanifold, let us now consider the existence of two almost Einstein structures on a conformal manifold, with a view to generalise to more than two. In[4], Gover & Leitner characterise the existence of two almost Einstein structures by the existence of conformal Killing vector fields.

Theorem 3 [Gover-Leitner]

If (M, c, σ_1) is an almost Einstein structure then (M, c, σ_2) is also an almost Einstein structure if and only if the vector field

$$k^a := \sigma_1 \nabla^a \sigma_2 - \sigma_2 \nabla^a \sigma_1$$

is a conformal Killing vector field. Moreover, for any Riemannoian space (M^d, g), $d \geq 2$ without boundary, admitting a (complete and essential[25]) conformal vector field then if (M, g) is non-compact then it is conformal to the Euclidean space $\mathbb{R}^d$. Likewise, if (M, g) is compact then it is conformal to the sphere S^d with round metric g_{rd}

These points shall be of use to us shortly. For now, let us examine the structure of the scale singularity sets; writing, for brevity, $\Sigma_K(\sigma) = \Sigma_K$ for some $K \in \mathcal{P} \backslash \{0\}$. A theorem of Gover & Leitner states

Theorem 4 [Gover & Leitner]
Let (M^d, c) be a Riemannian conformal space dimension $d \geq 3$. If $K_1, K_2 \in \mathcal{S}$ are linearly independent with $S_{K_2} < 0$ and $\Sigma_{K_2} \neq \emptyset$, then on $M \backslash \Sigma_{K_1}$, Σ_{K_2} (is totally umbillic and) has constant mean curvature with respect to the Einstein metric g determined by K_1. Moreover, Σ_{K_2} itself admits a directed almost Einstein structure provided by the part of K_1 orthogonal to K_2. That Σ_{K_2} is almost Einstein is due to the fact the tractor connection on (Σ_{K_2}, c_{K_2}) is a restriction of the ambient tractor connection to the orthogonal complement $K_2^{\perp}$ to K_2 in $\mathcal{S}$. Extensive comments on the restriction of the ambient tractor connection to scale singularity sets are given in[3]. In this instance, $\Sigma(\mathcal{S}) = \Sigma_{K_1} \cap \Sigma_{K_2}$ may either be empty or else is a totally umbilic hypersurface in (Σ_{K_2}, c_{K_2}) depending on the sign of K_1. It is given in[4] that $\Sigma(\mathcal{S})$ is a smooth hypersurface in (Σ_{K_2}, c_{K_2}) when K_1 is also scalar-negative. Indeed, that the intersection of scale singularities $\Sigma(\mathcal{S})$ is a totally umbillic hypersurface on conformal manifolds admitting multiple (≥ 2) scalar-negative almost Einstein structures was proved shortly thereafter in[21].

This in mind, let us now consider Leitner's *collapsing sphere product*. In so doing, we shall see that there is a reconstruction principle which enables one to identify the (conformal) boundary to a Poincaré-Einstein manifold with the scale singularity set of a particular conformal manifold that is conformally equivalent to a collapsing sphere product. Let $\overline{F}$ be a smooth $(m+1)$-manifold of dimension ≥ 3 with boundary $N := \partial F$ with an even[26] asymptotically hyperbolic metric g_+ on $F = \overline{F} \backslash N$. For $\mathbb{Z} \ni l \geq 0$, Leitner considers the product $S^l \times \overline{F}$ of the l-sphere S^l with $\overline{F}$, and remarks that the boundary

$$\partial(S^l \times \overline{F}) = S^l \times N.$$

Now, let

$$\Lambda : S^l \times \overline{F} \to \mathrm{Cl}(\overline{F}) \tag{7}$$

be the map which identifies the sphere S^l at each point of the boundary N to a single point. Leitner comments that the quotient space $\mathrm{Cl}(\overline{F})$ is a manifold without boundary of dimension $n := m + l + 1$, which is closed when $\overline{F}$ is compact. The image $\Lambda(S^l \times N)$ of identified points is a smooth codimension $l+1$ submanifold N_p of $\mathrm{Cl}(\overline{F})$. Importantly $\mathrm{Cl}(\overline{F})$ may be seen[21] to be diffeomorphic to $\partial(S^l \times F)$, N_p to the boundary $N = \partial F$, and a conformal structure $c[g_+]$ on induced $\mathrm{Cl}(\overline{F})$ from $\partial(S^l \times F)$. The pair $(\mathrm{Cl}(\overline{F}), c[g_+])$ is Leitner's *collapsing l-sphere product*[27]. The following is a remarkable reconstruction result given in[21]:

Theorem 5 [Leitner]
Let (M^n, c), $n \geq 3$ be a closed Riemannian conformal manifold admitting an Euclidean subspace $\mathcal{S} \subset \mathcal{P}$ of dimension $l > 1$ with $\Sigma(\mathcal{S}) \neq \emptyset$. If (M^n, c) is simply connected then it is globally conformally equivalent to the collapsing l-sphere product $(\mathrm{Cl}(\overline{F}), c[g_+])$ of some conformally compact, even Poincaré-Einstein space $(\overline{F}, g_+)$ of dimension $n - l$ such that the singularity set $\Sigma(\mathcal{S})$ corresponds to N_p.

IV. PROOF OF THE WITTEN-YAU CONJECTURE

We now have all necessary tools to begin our proof of the Witten-Yau conjecture.

Proof [Witten-Yau Conjecture]
Let $\overline{F}$ denote an $(m+1)$-dimensional Poincaré-Einstein manifold with boundary $\partial F = N$ on $F = \overline{F} \backslash N$. Consider the l-sphere product

$$S^l \times \overline{F}$$

as above, of dimension $n := m + l + 1$ with boundary $\partial(S^l \times \overline{F}) = S^l \times N$, itself of dimension $m + l$. Likewise, let the map Λ be as in (7) such that $(\mathrm{Cl}(\overline{F}), c[g_+])$ is the collapsing l-sphere product of $\overline{F}$. By theorem **5** we have that $(\mathrm{Cl}(\overline{F}), c[g_+])$ is conformally equivalent to some n-manifold M such that $\Sigma(\mathcal{S}) = N$ (upto diffeomorphism.) Now, since M possesses multiple almost Einstein structures, $\Sigma(\mathcal{S})$ inherits at least one from theorem **2**. Thereby, since N is without boundary (being itself a boundary), and possesses an almost Einstein structure, it is necessarily one of the manifolds from theorem **3**, which is to say that if N is non-compact then it is conformal to the Euclidean space $\mathbb{R}^d$ or if N is compact then it is conformal to the sphere S^d with round metric g_{rd}.

Both of these spaces admit metrics of non-negative scalar curvature. Now recall that the conjecture underlying the Witten-Yau theorem was precisely that the conformal boundary should admit a metric of non-negative scalar curvature for reasons pertaining to the stability of the CFT. Consequently, we have shown in generality that the conformal boundary N of any Poincaré-Einstein manifold $\overline{F}$ admits a metric of non-negative scalar curvature, which proves the conjecture. $\square$

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URL: http://www.encyclopediaofmath.org/index.php/Conformal_structure
[24] Unless the Witten-Yau theorem could be somehow avoided, that is.
[25] *Essential* conformal Killing vector fields are those which are not Killing vectors for any choice of metric in the conformal class
[26] *Even* in this context means that a special defining function for the metric, evaluated at the boundary, admits a power series expansion in terms of even degree, not that dim $N = 2k$ for $k \in \mathbb{Z}^+$.
[27] This construction is sometimes referred to as S^l-doubling.